

FINITE DIMENSIONAL APPROXIMATION PROPERTIES FOR TRACES AND CROSSED PRODUCTS

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ABSTRACT. In this note, we study the behavior of some finite dimensional approximation properties of traces on crossed products by actions of finite groups with Rokhlin property or tracial Rokhlin property. Let A be a simple separable unital C^* -algebra and let $\alpha : G \rightarrow \text{Aut}(A)$ be an action of a finite group G with tracial Rokhlin property. We show that every trace on $A \rtimes_\alpha G$ is quasidiagonal provided that all traces on A are uniformly quasidiagonal. Moreover, we prove that all traces on $A \rtimes_\alpha G$ are locally finite dimensional if we assume that A is a stably finite C^* -algebra whose all traces are uniformly locally finite dimensional and the order of projections on A is determined by traces.

Suppose that $\alpha : G \rightarrow \text{Aut}(A)$ is an action of a finite group G with Rokhlin property on a separable unital C^* -algebra A . Then we show that every trace on $A \rtimes_\alpha G$ is uniformly locally finite dimensional whenever all traces on A are uniformly locally finite dimensional.

1. INTRODUCTION

The purpose of this paper is to study the finite dimensional approximation properties of traces on crossed products of C^* -algebras whose all traces satisfying the certain finite dimensional approximation property, namely for uniformly quasidiagonality or uniformly locally finite dimensionality.

Group actions on C^* -algebras and their crossed products are one of the most central subjects in operator algebras. In particular, permanence properties for crossed products by actions of finite groups have been extensively studied. In this direction, Hirshberg and Winter [7], Phillips [14], Osaka and Phillips [13] and Pasnicu and Phillips [16] etc, investigated the structure of crossed products by actions of finite groups with Rokhlin property on unital C^* -algebras.

In [13], it was proved that if unital C^* -algebra A belongs to any of the following classes of C^* -algebras and α is an action of a finite group G with Rokhlin property, then $A \rtimes_\alpha G$ belongs to the same class:

- C^* -algebras with various kind of direct limit decomposition involving semi projective building block,
- simple unital AH -algebras with slow dimensional growth and real rank zero,
- C^* -algebra with real rank zero or stable rank one,
- simple C^* -algebras for which order on projections is determined by traces,

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- C^* -algebras satisfying UCT ,
- C^* -algebras with unique traces.

Phillips [14] showed that the above statement also holds for the class of AF -algebras. In addition, by [7] and [8], the following properties are also passing from an algebra to the crossed product by actions of finite groups with Rokhlin property:

- \mathcal{D} -stability where \mathcal{D} is a self-absorbing algebra,
- approximate divisibility,
- decomposition rank,
- nuclear dimension.

Rokhlin property is quite restrictive and it imposes some restriction on the relation between the K -theory of the original algebra, the action of the group on the K -theory, and the K -theory of the crossed products. If either the K_0 -group or K_1 -group of the algebra is isomorphic to \mathbb{Z} then the algebra has no action of a finite group with Rokhlin property. Indeed, Rokhlin property is not useful if there is no abundance of projections. A less restrictive version of Rokhlin property, "tracial Rokhlin property", was introduced by Phillips in [14]. There are relatively few actions with Rokhlin property and there are many algebras which admit no action with Rokhlin property. However, there are many examples of actions with tracial Rokhlin property, see [15].

Permanence properties of crossed products by actions of finite groups with tracial Rokhlin property were studied in [1], and [11]. In particular, in [11], it was proved that if A is a simple unital C^* -algebra, and that $\alpha : G \rightarrow \text{Aut}(A)$ is an action of a finite group with tracial Rokhlin property, then real rank zero, stable rank one or the property that the order on projections is determined by traces pass from A to $A \rtimes_\alpha G$.

Motivated by these results, we study the stability of some finite dimensional approximation properties of traces on unital C^* -algebras by taking crossed products of finite groups with Rokhlin property or tracial Rokhlin property. In particular, we obtain the following results.

Theorem 1.1. *Let A be a simple separable unital C^* -algebra and let $\alpha : G \rightarrow \text{Aut}(A)$ be an action of a finite group G with tracial Rokhlin property. Then we have the following.*

- (i) *Suppose that all traces on A are uniformly quasidiagonal then all traces on $A \rtimes_\alpha G$ are quasidiagonal.*
- (ii) *Suppose further that A is stably finite with real rank zero such that the order on projections is determined by traces. Then all traces on $A \rtimes_\alpha G$ are locally finite dimensional provided that all traces on A are uniformly locally finite dimensional.*

Theorem 1.2. *Suppose that A is separable unital C^* -algebra whose traces on A are uniformly locally finite dimensional and $\alpha : G \rightarrow \text{Aut}(A)$ is an action of a finite group G with Rokhlin property. Then all traces on $A \rtimes_\alpha G$ are uniformly locally finite dimensional.*

To study the finite dimensional approximation properties of traces on the crossed products of C^* -algebras whose traces are uniformly quasidiagonal, we follow the technique used in the section 8 of [2]. Let \mathcal{C}_{qd} be the class of unital C^* -algebras for which all traces are

quasidiagonal. By Proposition 8.3 of [2], if A is a simple, separable, unital C^* -algebra that is in TAC_{qd} , then all traces on A are quasidiagonal. The concept of TRC -algebras for a class \mathcal{C} of separable unital C^* -algebras was introduced by Elliott and Niu in [6]. Motivated by [2], we first study the behavior of TRC -algebras by taking crossed product by actions of finite groups with tracial Rokhlin property. Then we consider the special case of $\mathcal{C}_{u,qd}$, the class of all C^* -algebras whose traces are uniformly quasidiagonal.

To investigate the finite dimensional approximation properties of traces on crossed products of C^* -algebras whose traces are uniformly locally finite dimensional by actions of finite groups with tracial Rokhlin property, we essentially follow the idea used in the case of quasidiagonal traces. To do this, we first modify the notion of TRC -algebras and define $TRAC$ -algebras for a class \mathcal{C} of separable unital C^* -algebras. We compare these two notions and obtain some conditions on a C^* -algebra A which imply that A is a $TRAC$ -algebra if and only if A is a TRC -algebra.

The crucial fact to obtain our desired result is to prove that a unital separable $TRAC_{l,f,d}$ -algebra is a $TRAC_{l,f,d}$ -algebra, where $\mathcal{C}_{l,f,d}$ is the class of all unital separable C^* -algebras whose traces are locally finite dimensional.

Our approach to the case of crossed products by actions of finite groups with Rokhlin property on C^* -algebras whose traces are uniformly locally finite dimensional is inspired by the ideas employed in [13]. It is proved in [13] that some number of classes of separable unital C^* -algebras are closed under taking crossed product by actions of finite groups with Rokhlin property. The key observation in [13] is that in the case of actions of finite groups with Rokhlin property, the crossed product $A \rtimes_\alpha G$ has a local approximation property by C^* -algebras which are stably isomorphic to homomorphic images of A . This leads us to tackle our problem by proving the following: Suppose that a unital separable C^* -algebra A have some "local approximation property" (in the sense of Definition (2.10)) by the C^* -algebras whose traces are uniformly locally finite dimensional. Then every trace on A is uniformly locally finite dimensional.

This paper is organized as follows. Section 1 is devoted to study C^* -algebras whose traces are (uniformly) quasidiagonal or (uniformly) locally finite dimensional. In section 2, we first treat the crossed products of C^* -algebras for which all traces are uniformly quasidiagonal. Then we prove that all traces on the crossed products of C^* -algebras for which all traces are uniformly locally finite dimensional by actions of finite groups with Rokhlin property are locally finite dimensional. Moreover, we introduce and study the notion of $TRAC$ -algebras for a class \mathcal{C} of separable unital C^* -algebras. Then use this notion to study the crossed products of C^* -algebras whose traces are uniformly locally finite dimensional by actions of finite groups with tracial Rokhlin property.

2. FINITE DIMENSIONAL APPROXIMATION PROPERTIES FOR TRACES

In this section, we study the class of all unital separable C^* -algebras whose traces are (uniformly) quasidiagonal and the class of all separable unital C^* -algebras whose traces are (uniformly) locally finite dimensional.

A trace τ on A is called amenable if there exists a state ϕ on $B(H)$ such that $\phi|_A = \tau$ and $\phi(uTu^*) = \phi(T)$ for every unitary $u \in A$ and $T \in B(H)$. Amenable traces are satisfying in some form of natural finite dimensional approximation property: Let τ be a trace on a unital C^* -algebra A . Then by Theorem 3.1.6 of [3], τ is amenable if and only if there exists a sequence of u.c.p. maps $\phi_n : A \rightarrow M_{k(n)}$ such that $\|\phi_n(ab) - \phi_n(a)\phi_n(b)\|_2 \rightarrow 0$ and $\|\tau(a) - \text{tr}_{k(n)}\phi_n(a)\| \rightarrow 0$, for all $a \in A$. Moreover, we say that τ is uniformly amenable if we have $\|\text{tr}_{k(n)}\phi_n - \tau\|_{A^*} \rightarrow 0$. By Strengthening this property, Brown in [3] introduced and studied some subspaces of amenable traces: (uniform) quasidiagonal traces, (uniform) locally finite dimensional traces.

Definition 2.1. Let A be a separable C^* -algebra and τ be a trace on A .

(i) We say a trace τ is quasidiagonal if there exist completely positive contractions (c.c.p.) maps $\phi_n : A \rightarrow M_{k(n)}$ such that $\|\text{tr}_{k(n)}\phi_n(a) - \tau(a)\| \rightarrow 0$, for all $a \in A$ and $\|\phi_n(ab) - \phi_n(a)\phi_n(b)\| \rightarrow 0$ for all $a, b \in A$. Moreover, τ will be called uniformly quasidiagonal if one can further arrange that $\|\text{tr}_{k(n)}\phi_n - \tau\|_{A^*} \rightarrow 0$.

(ii) We say a trace τ is locally finite dimensional if there exist c.c.p. maps $\phi_n : A \rightarrow M_{k(n)}$ such that $\|\text{tr}_{k(n)}\phi_n(a) - \tau(a)\| \rightarrow 0$, for all $a \in A$ and $d(a, A_{\phi_n}) \rightarrow 0$ for all $a, b \in A$, where A_{ϕ_n} is the multiplicative domain of ϕ_n . Moreover, τ will be called uniformly locally finite dimensional if one can further arrange that $\|\text{tr}_{k(n)}\phi_n - \tau\|_{A^*} \rightarrow 0$.

Suppose that A is unital, then maps ϕ_n can be taken to be unital and completely positive (u.c.p.).

In the following remark we collect some classes of C^* -algebras whose traces are (uniformly) quasidiagonal.

Remark 2.2.

- Let A be a separable unital C^* -algebra with finite decomposition rank then all traces on A are quasidiagonal (Corollary 8.7, [2]).
- Every faithful trace on a nuclear quasidiagonal C^* -algebra A satisfying UCT is quasidiagonal (Theorem A, [17]).
- Let A be a nuclear C^* -algebra with unique trace τ , then τ is uniformly quasidiagonal (Theorem 6.1.3, [3]).

In this paper, we denote by \mathcal{C}_{qd} (resp. $\mathcal{C}_{u,qd}$) the class of all separable unital C^* -algebras whose traces are quasidiagonal (resp. uniformly quasidiagonal). In the following we investigate some properties of classes \mathcal{C}_{qd} and $\mathcal{C}_{u,qd}$.

Proposition 2.3. *Let A be a C^* -algebra that belongs to \mathcal{C}_{qd} . Then all traces on $C_0((0, 1]) \otimes A$ are quasidiagonal.*

Proof. Note that every trace on $C_0((0, 1]) \otimes A$ lies in the weak $*$ -closed convex hull of the set of traces in the form $\delta_t \times \tau$, where δ_t is the evaluation map at point t and τ is trace on B . Moreover by Proposition 3.5.1 of [3], the set of quasidiagonal traces on a C^* -algebra is a weak $*$ -closed convex set. Since all traces on A are quasidiagonal so Proposition 3.5.7 of [3] implies that all traces of the form $\delta_t \otimes \tau$ are quasidiagonal, thus we can conclude that all traces on $C_0((0, 1]) \otimes A$ are quasidiagonal. \square

Lemma 2.4. *Suppose that all traces on A are amenable. Then all traces on $C_0(0, 1] \otimes A$ are quasidiagonal.*

Proof. Note that by Proposition 3.5.1 of [3], the set of all amenable traces on a C^* -algebra is a weak $*$ -closed convex set together. Therefore part (1) of Proposition 3.7 of [3] enable us to use the similar argument given in the proof of Proposition (2.3) to conclude that all traces on $C_0(0, 1] \otimes A$ are amenable whenever all traces on A are amenable. Note that by Proposition 3.2 of [4], every amenable trace on a cone of a C^* -algebra is quasidiagonal, and so this completes the proof. \square

Corollary 2.5. *Let A be a C^* -algebra with the WEP. Then all traces on $C_0(0, 1] \otimes A$ are quasidiagonal.*

Proof. By Proposition 4.2.2 of [3], all traces on A are amenable. Hence Corollary (2.4) implies the result. \square

We recall from [13] the definition of a finitely saturated class of separable unital C^* -algebras.

Definition 2.6. Let \mathcal{C} be a class of separable unital C^* -algebras. Then we call \mathcal{C} finitely saturated if the following closure conditions hold:

- (1) If $A \in \mathcal{C}$ and $B \simeq A$, then $B \in \mathcal{C}$.
- (2) If $A \in \mathcal{C}$ and any integer n , then $M_n(A) \in \mathcal{C}$.
- (3) If $A \in \mathcal{C}$ and $p \in A$ is a nonzero projection, then $pAp \in \mathcal{C}$.
- (4) If A_1, A_2, \dots, A_n are in \mathcal{C} then $A_1 \oplus \dots \oplus A_n$ is in \mathcal{C} .

We say that \mathcal{C} is weakly finitely saturated if the conditions (1)-(3) hold.

Moreover, the finite saturation of a class \mathcal{C} is the smallest finitely saturated class which contains \mathcal{C} .

Proposition 2.7. *(due to N. Brown) Let τ be a uniformly quasidiagonal trace on A and p be a projection of A such that $\tau(p)$ is non zero, then $\frac{1}{\tau(p)}\tau$ restricts to a uniformly quasidiagonal trace on pAp .*

Proof. Suppose that $\phi_n : A \rightarrow M_{k(n)}$ are the u.c.p maps realizing the uniform quasidiagonality of τ . Since ϕ_n are asymptotically multiplicative, the positive operators $\phi_n(p)$ satisfy $\|\phi_n(p) - \phi_n(p)^2\| \rightarrow 0$ By functional calculus we can therefore find projections $P_n \in M_{k(n)}(\mathbb{C})$ such that $\|\phi_n(p) - P_n\| \rightarrow 0$. We claim that the maps which prove that $\frac{1}{\tau(p)}\tau$ restricts to a uniformly quasidiagonal trace on pAp are given by $\varphi_n(pap) = P_n\phi_n(pap)P_n$. In the other words, we will show

- (1) $\varphi_n(\cdot)$ are asymptotically multiplicative;
- (2) and for every $\varepsilon \geq 0$, there exists n such that $|\frac{1}{\tau(p)}\tau(pxp) - \frac{1}{tr(P_n)}tr(\varphi_n(pxp))| \leq \varepsilon$ for all contractions $x \in A$.

Both of these assertions require the following lemma.

Lemma 2.8. *For every $\epsilon \geq 0$, there exists n such that $\|P_n\phi_n(pxp) - \phi_n(pxp)\| \leq \epsilon$.*

Proof. Note that by Lemma 3.5 of [9], we have

$$\|\phi_n(px) - \phi_n(p)\phi_n(px)\| \leq \|\phi_n(p) - \phi_n(p)^2\|^{\frac{1}{2}}$$

for all contractions $x \in A$. Since $\|\phi_n(p) - \phi_n(p)^2\| \rightarrow 0$ and $\|\phi_n(p) - P_n\| \rightarrow 0$, it follows that for every $\epsilon \geq 0$, there exists n such that $\|\phi_n(px) - P_n\phi_n(px)\| \leq \epsilon$ for all contractions $x \in A$. Taking adjoint we get the same inequalities with p and P_n on the right side of x , and so some standard estimates complete the proof. \square

With this lemma we verify (1):

$$\varphi_n(pxppyp) = P_n\phi_n(pxppyp)P_n \approx \phi_n(pxppyp) \approx \phi_n(pxp)\phi_n(ppy) \approx \varphi_n(pxp)\varphi_n(ppy).$$

To verify (2) one observes that

$$\left| \frac{1}{\tau(p)}\tau(pxp) - \frac{1}{\text{tr}(P_n)}\text{tr}(\varphi_n(pxp)) \right|$$

is bounded above by the sum of

$$\left| \frac{1}{\tau(p)}\tau(pxp) - \frac{1}{\text{tr}(P_n)}\text{tr}(\phi_n(pxp)) \right|$$

and

$$\left| \frac{1}{\text{tr}(P_n)}\text{tr}(\phi_n(pxp)) - \frac{1}{\text{tr}(P_n)}\text{tr}(\varphi_n(pxp)) \right|.$$

Now, it is easy to see (2), and this completes the proof. \square

The following lemma can be concluded from Proposition (2.7) and Proposition 3.7 of [3].

Lemma 2.9. *The class $\mathcal{C}_{u.qd}$ is finitely saturated.*

Theorem 3.2 of [13] is a technical step in [13] to show that a number of classes of separable unital C^* -algebras are closed under taking crossed products by actions of finite groups with Rokhlin property. To recall this theorem, let us give the following definition.

Definition 2.10. Let \mathcal{C} be a class of separable unital C^* -algebras. A unital injective local \mathcal{C} -algebra is a separable unital C^* -algebra A with unit 1_A such that for every finite set $S \subseteq A$ and every $\epsilon > 0$, there is a C^* -subalgebra $1_A \in B \subseteq A$ in the finite saturation of \mathcal{C} such that $S \subseteq_\epsilon B$.

Let \mathcal{C} be a finitely saturated class of separable unital C^* -algebras, A be a C^* -algebra in \mathcal{C} and let $\alpha : G \rightarrow \text{Aut}(A)$ be an action of a finite group G with Rokhlin property. Suppose that finite subset $F \subseteq A$ and $\epsilon \geq 0$ are given. Then Theorem 3.2 of [13] states that there exists a projection f in A and a unital $*$ -homomorphism $\phi : M_{|G|}(\mathbb{C}) \otimes fAf \rightarrow A \rtimes_\alpha G$ such that $F \subseteq_\epsilon \phi(M_{|G|}(\mathbb{C}) \otimes fAf)$. Indeed by the proof, one can choose ϕ to be injective. Therefore we can conclude the following lemma.

Lemma 2.11. *Suppose that \mathcal{C} is a class of separable unital C^* -algebras. Let A be a C^* -algebra in \mathcal{C} and $\alpha : G \rightarrow \text{Aut}(A)$ be an action of a finite group G with Rokhlin property. Then $A \rtimes_\alpha G$ is a unital injective local \mathcal{C} -algebra.*

Theorem 3.2 of [13] enables authors to prove that some properties such as stable rank one, real rank zero, having unique trace are stable under taking crossed products by actions of finite groups with Rokhlin property. We will follow the same idea to investigate the finite dimensional approximation properties for traces on crossed products of C^* -algebras by actions of finite groups with Rokhlin property. To do this, we study the traces on injective local \mathcal{C} -algebras, where \mathcal{C} is the class of C^* -algebras whose traces are uniformly quasidiagonal or whose traces are uniformly locally finite dimensional. We first deal with the case of the class $\mathcal{C}_{u.qd}$.

Proposition 2.12. *Assume that A is a unital injective local \mathcal{C}_{qd} -algebra. Then all traces on A are quasidiagonal.*

Proof. Let τ be a trace on a unital injective local \mathcal{C}_{qd} -algebra A , and let a finite set $F = \{a_1, \dots, a_n\}$ of A and $\epsilon > 0$ be given. Assume that $\epsilon \leq 1$ and that F is a subset of the unit ball of A . Consider the finite set $S = F \cup \{a_i a_j, i, j = 1, \dots, n\} \subseteq A$. Then by our assumption on A for the finite set S and $\epsilon > 0$, there is a C^* -subalgebra $B \in \mathcal{C}_{qd}$ and there is a finite set $\tilde{S} = \{b_i, b_{jk}, i, j, k = 1, \dots, n\} \subseteq B$ such that $\|a_i - b_j\| \leq \epsilon$ and $\|a_i a_j - b_{ij}\| \leq \epsilon$, for all $i, j = 1, \dots, n$. Since the units of A and B are same, the restriction of τ on B defines a trace on B , denoted by $\hat{\tau}$. Thus $\hat{\tau}$ is a quasidiagonal trace on B and so there exists a u.c.p map $\varphi : B \rightarrow M_d$ such that $\|\varphi(xy) - \varphi(x)\varphi(y)\| \leq \epsilon$ and $\|tr_d \circ \varphi(x) - \tau(x)\| \leq \epsilon$, for all $x, y \in \tilde{S}$.

By Arveson's extension theorem, we can get a u.c.p map $\theta : A \rightarrow M_d$ extending φ . To continue, we need the following observation:

$$\|b_{ij} - b_i b_j\| \leq \|b_{ij} - a_i a_j\| + \|a_i a_j - a_i b_j\| + \|a_i b_j - b_i b_j\| \leq 4\epsilon.$$

For the last inequality we use this fact that $\|b_i\| \leq \|b_i - a_i\| + \|a_i\| \leq \epsilon + 1 \leq 2$. Hence, we can compute

$$\begin{aligned} \|\theta(a_i a_j) - \theta(a_i) \theta(a_j)\| &\leq \|\theta(a_i a_j) - \varphi(b_{ij})\| + \|\varphi(b_{ij}) - \varphi(b_i b_j)\| + \\ &\|\varphi(b_i b_j) - \varphi(b_i) \theta(a_j)\| + \|\varphi(b_i) \theta(a_j) - \theta(a_i) \theta(a_j)\| \leq 10\epsilon. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \|tr_d \circ \theta(a_i) - \tau(a_i)\| &\leq \|tr_d \circ \theta(a_i) - tr_d \circ \varphi(b_i)\| + \\ &\|tr_d \circ \varphi(b_i) - \tau(b_i)\| + \|\tau(b_i) - \tau(a_i)\| \leq 3\epsilon. \end{aligned}$$

Therefore, we have shown that τ is a quasidiagonal trace on A , as desired. \square

Brown in [3] discussed that locally finite dimensional traces can play an important role in the Elliott classification program of C^* -algebras. In particular, the author gave a simple characterization of tracially AF -algebras in terms of tracial approximation properties of traces.

Lin in [10], introduced the tracially AF -algebras, inspired by Popa's algebras and the classification theory of C^* -algebras. Then he proved that all simple C^* -algebras of real rank zero which are classified in [5] are tracially AF -algebras. We recall the following result from [14], we will refer to it later.

Proposition 2.13 (Proposition 2.3 of [14]). *Let A be a simple separable unital C^* -algebra. Then A is a tracially AF -algebra if and only if for any $\epsilon > 0$, any finite subset $F \subseteq A$, and any non-zero $a \in A_+$, there exist a non-zero projection $p \in A$ and a finite-dimensional C^* -subalgebra $C \subseteq A$ such that $1_C = p$, and for all $x \in F$,*

- (1) $\|xp - px\| \leq \epsilon$,
- (2) $pxp \subseteq_\epsilon C$,
- (3) $1 - p$ is Murray-von Neumann equivalent to a projection in \overline{aAa} .

In this paper, we denote by $\mathcal{C}_{l.f.d.}$ (resp. $\mathcal{C}_{u.l.f.d.}$) the class of all separable unital C^* -algebras whose traces are locally finite dimensional (resp. uniformly locally finite dimensional).

In the following remark, we give examples of C^* -algebras which belong to the class $\mathcal{C}_{u.l.f.d.}$.

Remark 2.14 ([3]). Suppose that A is a separable unital C^* -algebra. Then each of the following conditions implies that $A \in \mathcal{C}_{u.l.f.d.}$.

- (1) A is a type one C^* -algebra.
- (2) A is a tracially AF -algebra.
- (3) A has real rank zero and finite decomposition rank.
- (4) A is simple and quasidiagonal with unique trace.

We end this section with the behavior of traces on unital injective local $\mathcal{C}_{u.l.f.d.}$ -algebras.

Lemma 2.15. *The class $\mathcal{C}_{u.l.f.d.}$ is finitely saturated.*

Proof. It follows immediately by Proposition 3.7.5 and Lemma 4.5.3 of [3]. □

Lemma 2.16. *Assume that A is a unital injective local $\mathcal{C}_{u.l.f.d.}$ -algebra. Then all traces on A are uniformly locally finite dimensional.*

Proof. Suppose that A is a unital injective local $\mathcal{C}_{u.l.f.d.}$ -algebra and τ is a trace on A . Let finite set F of A and $\epsilon \geq 0$ be given. So there is a C^* -subalgebra C of A in $\mathcal{C}_{u.l.f.d.}$ and a finite set $S \subseteq C$ such that for every $x \in F$ there is $y_x \in S$ with $\|x - y_x\| \leq \frac{\epsilon}{2}$. Since C and A has the same unit, $\tau|_C$ is a trace on C . Hence there is a u.c.p map $\phi : C \rightarrow M_k$ such that $d(S, C_\phi) \leq \frac{\epsilon}{2}$ and $\|\tau|_C - tr_k \otimes \phi\|_{C^*} \leq \epsilon$. Note that $d(F, C_\phi) \leq \epsilon$. Now use Lemma 4.4.1 of [3] to conclude that τ is uniformly locally finite dimensional. □

3. CROSSED PRODUCTS

In this section, we study the finite dimensional approximation properties for traces on $A \rtimes_\alpha G$ when A is a separable unital C^* -algebra belonging to the class $\mathcal{C}_{u.qd}$ or the class $\mathcal{C}_{u.l.f.d.}$, and $\alpha : G \rightarrow \text{Aut}(A)$ is an action of a finite group G with tracial Rokhlin property or Rokhlin property. Let us to begin this section with recalling the Izumi's definition of actions with Rokhlin property.

Definition 3.1. Let A be an infinite dimensional separable unital C^* -algebra, and let $\alpha : G \rightarrow \text{Aut}(A)$ be an action of a finite group G on A . We say that α has the Rokhlin property if for every finite set $F \subseteq A$, every $\epsilon \geq 0$, there are mutually orthogonal projections $e_g \in A$ for every $g \in G$ such that:

- (1) $\|e_g a - a e_g\| \leq \epsilon$,
- (2) $\|\alpha_g(e_h) - e_{gh}\| \leq \epsilon$,
- (3) $1 = \sum_{g \in G} e_g$.

Phillips in [14] gives the definition of tracial Rokhlin property. The difference is that one does not require that $1 = \sum_{g \in G} e_g$, only that $1 - \sum_{g \in G} e_g$ is small in tracial sense.

Definition 3.2. Let A be an infinite dimensional simple separable unital C^* -algebra, and let $\alpha : G \rightarrow \text{Aut}(A)$ be an action of a finite group G on A . We say that α has the tracial Rokhlin property if for every finite set $F \subseteq A$, every $\epsilon > 0$, and every positive element $x \in A$ with $\|x\| = 1$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

- (1) $\|e_g a - a e_g\| \leq \epsilon$,
- (2) $\|\alpha_g(e_h) - e_{gh}\| \leq \epsilon$,
- (3) With $e = \sum_{g \in G} e_g$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x .
- (4) With e in (3), we have $\|exe\| \geq 1 - \epsilon$.

3.1. Uniformly quasidiagonal traces. We first consider the case of crossed products by actions of finite groups with Rokhlin property on C^* -algebras belonging to $\mathcal{C}_{u.qd}$.

Proposition 3.3. *Let A be a unital C^* -algebra in $\mathcal{C}_{u.qd}$ and let $\alpha : G \rightarrow \text{Aut}(A)$ be an action with Rokhlin property. Then every trace on $A \rtimes_\alpha G$ is quasidiagonal.*

Proof. Note that by Lemma (2.9) and Lemma (2.11), $A \rtimes_\alpha G$ is an injective unital local \mathcal{C}_{qd} -algebra. Thus apply Proposition (2.12) to deduce that all traces on $A \rtimes_\alpha G$ are quasidiagonal. \square

To study the case of traces on crossed products by actions of finite groups with tracial Rokhlin property, we use the notion of tracially approximated C^* -algebras by a class of separable unital C^* -algebras. We recall the following definition from [6].

Definition 3.4. Let \mathcal{C} be a class of separable unital C^* -algebras. The class of unital C^* -algebras which are tracially approximated by C^* -algebras in \mathcal{C} , denoted by TAC , is defined as follows. A unital C^* -algebra A is said to belong to the class TAC if for any $\epsilon > 0$, any finite subset $F \subseteq A$, and any non-zero $a \in A_+$, there exist a non-zero projection $p \in A$ and a C^* -subalgebra $C \subseteq A$ such that $C \in \mathcal{C}$, $1_C = p$, and for all $x \in F$,

- (i) $\|xp - px\| \leq \epsilon$,
- (ii) $pxp \subseteq_\epsilon C$,
- (iii) $1 - p$ is Murray-von Neumann equivalent to a projection in \overline{aAa} .

In the following propositions, we investigate some properties of TRC -algebras under taking crossed products by finite groups.

Proposition 3.5. *Let \mathcal{C} be a weakly finitely saturated class of separable unital C^* -algebras and A be a simple separable unital C^* -algebra in TAC . Suppose that α is an action of finite group G on A with tracial Rokhlin property but not Rokhlin property. Then $A \rtimes_\alpha G$ is in TAC .*

Proof. By Lemma 1.13 of [14], α has Rokhlin property or A has property (SP) . Thus we can assume that A has property (SP) . Observe that Lemma 2.3 of [6] and the assumption that \mathcal{C} is weakly finitely saturated imply that the class TAC is closed under taking tensoring with matrix algebras as well as taking hereditary C^* -subalgebras. This observation enables us to conclude the proposition by a similar argument given in the proof of Theorem 2.6 of [14]. \square

We remark here that a similar result to Proposition (3.5) was proved in Theorem 3.3 of [11] with different proof.

Now, we turn our attention to the class \mathcal{C}_{qd} . Before stating our main result in this subsection, we recall the next proposition from [2].

Proposition 3.6. *(Proposition 8.3, [2]) Let A be a simple separable unital C^* -algebra in TAC_{qd} . Then $A \in \mathcal{C}_{qd}$.*

The following result can be concluded by Proposition 8.3 of [2], Lemma (2.9) and Proposition (3.5).

Corollary 3.7. *Let A be a simple separable unital C^* -algebras in $\mathcal{C}_{u,qd}$ and let α be an action of a finite group G on A with tracial Rokhlin property. Then all traces on $A \rtimes_\alpha G$ are quasidiagonal.*

3.2. Uniformly locally finite dimensional traces. In this subsection, we study the finite-dimensional approximation properties of traces on crossed product $A \rtimes_\alpha G$ where A is a C^* -algebra in $\mathcal{C}_{u,l,f,d}$ and α is an action of a finite group G with Rokhlin property or tracial Rokhlin property.

First we deal with the case of actions with Rokhlin property.

Theorem 3.8. *Let A be a unital separable C^* -algebra in $\mathcal{C}_{u,l,f,d}$ and $\alpha : G \rightarrow \text{Aut}(A)$ be an action of a finite group G with Rokhlin property. Then all traces on $A \rtimes_\alpha G$ are uniformly locally finite dimensional.*

Proof. By Lemma (2.15), $\mathcal{C}_{u,l,f,d}$ is finitely saturated, hence Lemma (2.11) implies that $A \rtimes_\alpha G$ is an injective local $\mathcal{C}_{u,l,f,d}$ -algebra. Now by Lemma (2.16), all traces on $A \rtimes_\alpha G$ are uniformly locally finite dimensional, as desired. \square

It follows from the proof of Proposition 4.5.5 of [3] that all traces on a tracially AF -algebra are uniformly locally finite dimensional. Moreover, Theorem 4.5.1 of [3] states that a simple C^* -algebra is tracially AF -algebra if and only if A has stable rank one, real rank zero, weakly unperforated K -theory such that every finite subset $F \subseteq A$ and $\epsilon > 0$ there exists a finite dimensional subalgebra $B \subseteq A$ with unit e such that for all $x \in F$ and $\tau \in T(A)$ we have

- (i) $\|xe - ex\| \leq \epsilon$,
- (ii) $exe \subseteq_\epsilon C$,
- (iii) $\tau(e) \geq 1 - \epsilon$.

These results from [3] motivate us to give the following definition to study uniform locally finite dimensional traces on C^* -algebras.

Definition 3.9. Let \mathcal{C} be a class of separable unital C^* -algebras. A unital C^* -algebra A is called a *TRAC*-algebra if for any finite set $F \subseteq A$ and $\epsilon \geq 0$, there exists a C^* -algebra $C \in \mathcal{C}$ with unit e such that for all $x \in F$ and every trace τ on A we have

- (i) $\|xe - ex\| \leq \epsilon$,
- (ii) $exe \subseteq_\epsilon C$,
- (iii) $\tau(e) \geq 1 - \epsilon$.

In the same sprite as in Lemma 1.4 of [12], we can obtain the following result. For the convenience of the reader, we prove it.

Proposition 3.10. *Let \mathcal{C} be a class of separable unital C^* -algebras. Suppose that A is a simple stably finite real rank zero C^* -algebra with property (SP) such that the order on projections is determined by traces. Then A is a TRAC-algebra if and only if A is a TRC-algebra.*

Proof. First assume that A belongs to the class TRC , and finite set F of A and $\epsilon \geq 0$ are given. Then by the proof of the Lemma 1.4 of [12], there exists a non-zero projection $x \in A$ such that $\tau(x) \geq \epsilon$ for all traces τ on A . Now, use definition (3.4) to F , ϵ and x to find a C^* -subalgebra B of A in \mathcal{C} with unit p . Conversely assume that A is in the class of *TRAC*, and let a finite set F of A , $\epsilon \geq 0$ and a nonzero positive element x in A be given. Since A has property (SP), there exists a non zero projection $q \in \overline{xAx}$. Now apply Definition (3.2) to F and $\epsilon = \min\{\epsilon, \inf_{\tau \in T(A)} \tau(q)\}$ to find a C^* -subalgebra B of A in the class \mathcal{C} with unit p satisfying the conditions of the definition. Then use the property of A that its order on projection is determined by traces to conclude that $1 - p \preceq q$ \square

The following theorem is essential to obtain our main result in this subsection.

Theorem 3.11. *Suppose that a separable unital C^* -algebra A belongs to $TRAC_{u.l.f.d.}$. Then all traces on A are locally finite dimensional.*

Proof. Let τ be a trace on A , $F = \{a_1, \dots, a_n\}$ be a finite set of A and let $\epsilon \geq 0$ be given. Since $A \in TRAC_{l.f.d.}$, there is a C^* -algebra $C \in \mathcal{C}$ with unit e which satisfies the definition (3.2) for F , $\frac{\epsilon}{4}$ and $\mathcal{C}_{u.l.f.d.}$. We can assume that $F \subseteq A_+$ and each a_i is in the unit ball of A . By condition (ii) of the definition (3.2), for each $1 \leq j \leq n$, there is $c_j \in C$ such that $\|ea_j e - c_j\| \leq \frac{\epsilon}{4}$. Note that $\frac{1}{\tau(e)}\tau(\cdot)$ is trace on C so there is a u.c.p map $\phi : C \rightarrow M_d$ such that for all $1 \leq j \leq n$ there is $d_j \in C_\phi$ satisfying $\|d_j - c_j\| \leq \epsilon$ and $\|tr_d \phi(d_j) - \tau(d_j)\| \leq \epsilon$. By condition (ii), B is subalgebra of eAe , so by Arveson extension theorem, we can extend ϕ to a u.c.p map $\bar{\phi}$ from eAe to M_d . Then compose this map with the conditional expectation $E : A \rightarrow eAe$ defined by $E(a) = eae$ to obtain a u.c.p map $\theta : A \rightarrow M_d$. We claim that u.c.p map θ is our desired map satisfying the definition

of locally finite dimensional traces for the given finite set F and $\epsilon \geq 0$. To see this, first note that by assumption $\|a_i e - e a_i\| \leq \frac{\epsilon}{4}$ and the choice of c_i , we have

$$\|a_i - c_i - e^\perp a e^\perp\| \leq \|a_i - e a_i e - e^\perp a e^\perp\| + \|e a_i e - e c_i e\| \leq \epsilon.$$

Clearly, $c_i + e^\perp a_i e^\perp$ is in the multiplicative domain of θ , so $F \subset_\epsilon C_\theta$. Moreover, we have

$$\|tr_d \theta(a_i) - \frac{1}{\tau(e)} \tau(c_i)\| = \|tr_d \bar{\phi}(e a_i e) - \frac{1}{\tau(e)} \tau(c_i)\| \leq \frac{\epsilon}{4} + \|tr_d \theta(a_i) - tr_d \phi(c_i)\| \leq \epsilon$$

Since $\tau(e)$ is close to one in norm by the assumption, we can see that $\frac{1}{\tau(e)} \tau(c_i)$ is close enough to $\tau(c_i)$ and so it is enough close to $\tau(a_i)$, as desired. \square

Corollary 3.12. *Let A be a stably finite simple separable unital C^* -algebra with real rank zero such that the order over projections on A is determined by traces. Suppose that all traces on A are uniformly locally finite dimensional and $\alpha : G \rightarrow \text{Aut}(A)$ is an action of a finite group G with tracial Rokhlin property. Then all traces on $A \rtimes_\alpha G$ are locally finite dimensional.*

Proof. It follows from [1] that the order over projection in crossed product $A \rtimes_\alpha G$ is determined by traces and it has real rank zero. Next, we show that $A \rtimes_\alpha G$ is stably finite. Note that $M_n(A \rtimes_\alpha G) \simeq M_n(A) \rtimes_{\alpha \otimes \iota} G$, so we need to show that $A \rtimes_\alpha G$ is finite. To see this, it is enough to view $A \rtimes_\alpha G$ as a unital C^* -subalgebra of $M_k(A)$, where k is the cardinality of finite group G , by $au_g \mapsto \sum_{h \in G} e_{h, g^{-1}h} \otimes \alpha_{h^{-1}}(a)$.

Moreover, by Proposition (3.5), $A \rtimes_\alpha G$ is a $TRC_{u.l.f.d}$ -algebra, and since $A \rtimes_\alpha G$ is stably finite, the traces separate the projections and it has real rank zero, we can employ Theorem (3.2) to conclude that $A \rtimes_\alpha G$ is a $TRAC_{u.l.f.d}$ -algebra. Therefore, by Theorem (3.11) every trace on $A \rtimes_\alpha G$ is locally finite dimensional. \square

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